PHY 341 HW Ch.2a

Do problems 2.4, 2.5; plus the following:

q2-1 Given the definitions of the lowering and raising operators,

$$a_{\pm} = \frac{\mp ip + m\omega x}{\sqrt{2\hbar m\omega}},$$

show that

$$a_+ a_- = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2},$$

where $\hat{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$.

q2-2

Let $\psi_1(x)$ and $\psi_2(x)$ be the normalized stationary states with energies E_1 and E_2, respectively. The wave function $\Psi(x, 0)$ at $t = 0$ is given by

$$\Psi(x, 0) = c_1\psi_1(x) + c_2\psi_2(x).$$

(a) If $c_1 = \frac{1}{2}$, find c_2 that makes $\Psi(x, 0)$ normalized, assuming c_2 is real and positive.
(b) Find the expectation value of energy. Does it depend on time for $t > 0$?
(c) Calculate the expectation value of position $\langle x\rangle$. Express your answer in terms of $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_{12}$, defined as (no need to evaluate)

$$\bar{x}_1 = \int \psi_1^* x \psi_1 \, dx, \quad \bar{x}_2 = \int \psi_2^* x \psi_2 \, dx, \quad \bar{x}_{12} = \int \psi_1^* x \psi_2 \, dx.$$

Does $\langle x\rangle$ depend on time for $t > 0$?

q2-3

Numerical exploration of Problem **q2-2**. Now assume the states are the first two states of a particle in a box. Use the same coefficients c_1 and c_2, write down of wave function $\Psi(x, t)$. Then follow the sample Jupyter code at `https://jwang.sites.umassd.edu/p341/` for superposition of states, plot the probability density $|\Psi(x, t)|^2$ at different times. Use atomic units (a.u.), where the constants $m = \hbar = a = 1$. Argue, based on the graph, that $\Psi(x, t)$ is not a stationary state, and the expectation value $\langle x\rangle$ depends on time.
For bonus, plot the real and imaginary parts of $\Psi(x, t)$. In Python this can be done as `Psi.real` and `Psi.imag`. Comment on your observations.
Extra credit for the fearless: Numerically check that the normalization $N = \int_0^a |\Psi(x, t)|^2 dx = 1$ holds for arbitrary t. Follow the sample code for normalization at the course link. Plot both N and $N - 1$ vs time on separate graphs.

You can submit the results and graphs, or email me the code if it contains animation.

q2-4

Let $f = f(x)$ be a differentiable function. By acting each commutator on a test wave function ψ, show that
(a) $[f, x] = 0$;
(b) $[f, p] = i\hbar f'$;
(c) $[x, fp] = i\hbar f$.